Mathematics Framework Solution Sets
Grade 7

A note about these solutions.

These solutions are intended for teachers, not students. The solutions are fairly detailed and some include additional comments that serve to further explain the content and purpose of each problem. It is important to note that these solutions are not meant to be representative of student solutions.

It is the nature of many mathematics problems that they can be solved in different ways. The solutions given here represent simply one way of solving the problems. At times, a second solution path is offered in the Further Explanation boxes.

It is our hope that these solution sets help teachers to better see the essential skills and concepts that are important to student success in Grade 7 mathematics.

A note about notation.

Throughout these solutions, points are represented by capital letters. Also, AB is used to denote the segment connecting points A and B, while $\overline{AB}$ is used to denote the length of segment AB. Therefore, we will write $AB \cong CD$ to mean that segment AB *is congruent to* CD, and $\overline{AB} = \overline{CD}$ to mean that they have the same length. (A necessary distinction.)

A similar convention holds for $\angle ABC$ and $m\angle ABC$, wherein the former denotes the angle itself and the latter denotes the measure of the angle.

Also, $\overleftrightarrow{AB}$ will be used to denote the unique line through the points A and B, of which the segment AB is a part. By a slight abuse of notation, we will write $AB||CD$ to mean that segments AB and CD are parallel (it is more appropriate to state AB and CD are parts of the parallel lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$).

Number Sense 1.1 Read, write, and compare rational numbers in scientific notation (positive and negative powers of 10), compare rational numbers in general.

Problem: Convert to scientific notation; compute and express your answer in scientific notation and in decimal notation.

1. $\dfrac{(350,000)(0.0049)}{0.25}$

2. $\dfrac{(0.000042)(0.0063)}{(140,000)(70,000)(0.18)}$

Solution: 1. First, express the numbers in the numerator and denominator in scientific notation:

$$350,000 = 3.5 \times 10^5, \quad 0.0049 = 4.9 \times 10^{-3}, \quad 0.25 = 2.5 \times 10^{-1}$$

Next, we rewrite the fraction as

$$\frac{(350,000)(0.0049)}{.25} = \frac{(3.5 \times 10^5)(4.9 \times 10^{-3})}{2.5 \times 10^{-1}}.$$

We can use the commutative and associative laws to rearrange factors to get

$$\begin{aligned}\frac{(3.5 \times 10^5)(4.9 \times 10^{-3})}{2.5 \times 10^{-1}} &= \frac{3.5 \times 4.9 \times 10^5 \times 10^{-3}}{2.5 \times 10^{-1}} \\ &= \frac{3.5 \times 4.9}{2.5} \times \frac{10^5 \times 10^{-3}}{10^{-1}}.\end{aligned}$$

Using rules of exponents, we have

$$\frac{10^5 \times 10^{-3}}{10^{-1}} = 10^{5+(-3)-(-1)} = 10^3,$$

while

$$\frac{3.5 \times 4.9}{2.5} = 6.856$$

Therefore,

$$\frac{(350,000)(0.0049)}{0.25} = 6.856 \times 10^3 = 6,856$$

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2. As in the previous problem, we have

$$\frac{(0.000042)(0.0063)}{(140,000)(70,000)(0.18)} = \frac{(4.2 \times 10^{-5})(6.3 \times 10^{-3})}{(1.4 \times 10^{5})(7 \times 10^{4})(1.8 \times 10^{-1})}$$
$$= \frac{4.2 \times 6.3 \times 10^{-5} \times 10^{-3}}{1.4 \times 7.0 \times 1.8 \times 10^{5} \times 10^{4} \times 10^{-1}}$$

Again, using laws of exponents, we have

$$\frac{10^{-5} \times 10^{-3}}{10^{5} \times 10^{4} \times 10^{-1}} = 10^{-5+-3-5-4-(-1)} = 10^{-16}.$$

Also,

$$\frac{4.2 \times 6.3}{1.4 \times 7.0 \times 1.8} = 1.5.$$

Thus,

$$\frac{(0.000042)(0.0063)}{(140,000)(70,000)(0.18)} = 1.5 \times 10^{-16}.$$

Further Explanation: To express a number n in scientific notation, we find a number a such that $a \times 10^{p} = n$, and $1 \leq a < 10$. Thus,

$$85,647 = 8.5647 \times 10^{4} \quad \text{and} \quad 0.0005874 = 5.874 \times 10^{-4}.$$

After writing numbers in scientific notation, we can simply work with all the powers of 10 using the usual rules of exponents.

Number Sense 1.6 Calculate the percentage of increases and decreases of a quantity.

Problem: Peter was interested in buying a basketball. By the time he saved enough money, everything in the sporting goods store had been marked up by 15%. Two weeks later, however, the same store had a sale, and everything was sold at a 15% discount. Peter immediately bought the ball, figuring that he was paying even less than before the prices were raised. Was he mistaken?

Solution: The question is asking whether the price of the ball is less now after the 15% discount. Since the price of the basketball is unknown, we'll let x represent the original price of the basketball in dollars. The price P of the ball after the first markup of 15% can be represented as

$$P = x + (0.15)x = 1.15x.$$

When the ball gets marked down 15%, this means the reduced price R is now $(100 - 15)\%$ or 85% of the price P:

$$R = 0.85P = 0.85(1.15x) = 0.9775x.$$

This shows that the reduced price R is 97.75% of the original price x, so that Peter was not mistaken. The ball costs less after the discount.

Number Sense 1.7 Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

Problem: What will be the monthly payments on a loan of $50,000 at 12% annual interest so that it will be paid off at the end of 10 years? How much total interest will have been paid? Do the same problem with 8% annual interest over 10 years. Do the same problem with 10% annual interest over 15 years. Solve the problem using simple interest. (Use calculators.)

Solution: Let P be the principle, r the rate, and t the number of years of the loan. The general formula for computing the value of the loan using *compound interest* in this case is

$$A = P(1+r)^t.$$

If the interest is *simple*, then the interest per year is Pr, so that the total interest is Prt.
Here, the principle is $P = \$50,000$.
$\underline{r = 12\%,\ t = 10}$
First suppose we had compound interest. Then

$$A = 50,000(1+0.12)^{10} = 155,292.41$$

is the total amount that must be paid off. In 10 years there are 120 months, so the monthly payment will be

$$\frac{\$155,292.41}{120} = \$1,294.10.$$

The total interest in this case is $\$155,292.41 - \$50,000 = \$105,292.41$.
If the interest is simple, then we have $\$50,000 \times 0.12 = \$6,000$ interest per year, or $\$60,000$ over ten years. Thus, the amount payed back is $\$110,000$. Over 120 months, this gives a monthly payment of $\dfrac{\$110,000}{120} = \$916.67$.
$\underline{r = 8\%,\ t = 10}$
As before, if compounded annually, we have

$$A = 50,000(1+0.08)^{10} = 107,946.25$$

The monthly payment is $\$899.55$ and the total interest is $\$57,946.25$.
If simple, then the interest per year is $\$50,000 \times 0.08 = \$4,000$, so that the total interest is $\$40,000$. The monthly payment is $\dfrac{\$90,000}{120} = \$750$.
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$r = 10\%, t = 15$

If compounded annually, we get

$$A = 50,000(1 + 0.1)^{15} = 208,862.41$$

The monthly payment is

$$\frac{\$208,862.41}{180} = \$1,160.35$$

since there are 180 months in the period. The total interest is $\$208,862.41 - \$50,000 = \$158,862.41$.

When using simple interest, we get yearly interest of $\$50,000 \times 0.10 = \$5,000$. Over 15 years, this gives $\$75,000$ in interest. The total amount paid is then $\$125,000$, so that the monthly payment is $\frac{\$125,000}{180} = \$694.44$.

Further Explanation: *Interest* is a fee (or payment) made on money that is borrowed (or loaned). *Simple interest* is interest paid only on the amount borrowed. It is computed using $I = Prt$ where I is the interest, P is the principal, r is the annual interest rate, and t is the time in years.

In *compound interest*, the interest is taken not only on the principal but also on the interest. Each time the interest is calculated, it is added to the principal, and interest is then taken on this new total amount. Let P represent the principle, r the annual percentage rate (as a decimal), t the number of years of the loan, and n the number of times per year the interest is calculated. Then the total loan amount can be found by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

For example, suppose you invest $1,000 at 8% compounded quarterly, and you'd like to compute what you will pay in one year. The *quarterly interest rate* is $r/n = 0.08 \div 4 = 0.02$ (since there are $n = 4$ periods in a year when interest is compounded quarterly). Here, $t = 1$, so that

$$A = (1,000)(1 + 0.02)^{4\cdot 1} = (1,000)(1.02)^4 = \$1,082.43.$$

We can easily find the total interest in this case, by subtracting

$$\$1,082.43 - \$1,000 = \$82.43$$

Number Sense 2.2 Add and subtract fractions by using factoring to find common denominators.

Problem: Reduce $\dfrac{910}{1,859}$ to lowest terms.

Solution: We apply the Fundamental Theorem of Arithmetic (FTA) to factor the numerator and denominator:

$$\frac{910}{1,859} = \frac{2 \times 5 \times 7 \times 13}{11 \times 13^2}.$$

Since $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$, we have

$$\frac{2 \times 5 \times 7 \times 13}{11 \times 13 \times 13} = \frac{2 \times 5 \times 7}{11 \times 13} \times \frac{13}{13}.$$

Finally, $\dfrac{a}{a} = 1$ for $a \neq 0$, so that

$$\frac{2 \times 5 \times 7}{11 \times 13} \times \frac{13}{13} = \frac{70}{143} \times 1 = \frac{70}{143}.$$

Further Explanation: Crucial to students' ability to reduce rational numbers and operate with fractions is an understanding of the Fundamental Theorem of Arithmetic (FTA). It states that every whole number $x > 1$ has a unique factorization into prime numbers

$$x = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k},$$

where the p_is are prime numbers and $n_i \geq 1$ for all i.

Number Sense 2.2 Add and subtract fractions by using factoring to find common denominators.

Problem: Subtract and reduce to lowest terms:

$$\frac{81}{143} - \frac{7}{208}.$$

Solution: To find a common denominator, we factor each denominator:

$$\begin{aligned} 143 &= 11 \times 13 \\ 208 &= 2 \times 2 \times 2 \times 2 \times 13 = 2^4 \times 13. \end{aligned}$$

The least common denominator (LCD) of the fractions is the smallest positive number that is a multiple of each of 143 and 208, also known as the Least Common Multiple (LCM) of 143 and 208. It can be found by taking the product of each prime factor appearing in the two factorizations above to the highest exponent with which they appear. Thus, the LCM of 143 and 208 is $2^4 \times 11 \times 13 = 16 \times 11 \times 13 = 2,288$. We also observe that $143 \times 16 = 2,288$ and $208 \times 11 = 2,288$.

We convert each fraction into an equivalent one with the common denominator, and subtract the numerators:

$$\begin{aligned} \frac{81}{143} - \frac{7}{208} &= \frac{81 \times 16}{143 \times 16} - \frac{7 \times 11}{208 \times 11} \\ &= \frac{1,296}{2,288} - \frac{77}{2,288} \\ &= \frac{1,296 - 77}{2,288} \\ &= \frac{1,219}{2,288} \end{aligned}$$

The only way this fraction could be reduced is if 2, 11 or 13 was a factor of $1,219$. Clearly $1,219$ is not even, so 2 is not a factor. Since $11 \times 111 = 1,221$ and this is 2 more than $1,219$, 11 is not a factor either. Finally, since $13 \times 93 = 1,209$, which is 10 less than 1,219, we see that 13 is not a factor either.

Further Explanation: If n is a multiple of m, then the prime factorization of n must contain all of the prime factors of m to an exponent greater than or equal to their exponent in the factorization for m. For example, clearly 1800 is a multiple of 60, and $1800 = 2^3 \cdot 3^2 \cdot 5^2$ while $60 = 2^2 \cdot 3 \cdot 5$.

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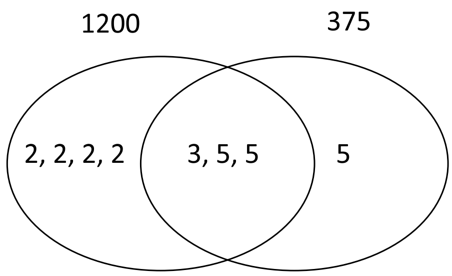
This idea leads to the method for finding the Least Common Multiple of two positive numbers x and y that is used above. The *least common multiple* of x and y is the smallest positive number L that is a multiple of both x and y.
For example, the LCM of 40 and 25 is $200 = 5 \cdot 40 = 8 \cdot 25$. Other common multiples of 40 and 25 are 400, 600, and 800; notice that each is a multiple of the LCM 200 as well.
Since the LCM is a multiple of both x and y, it must contain the prime factors of *each* number to an exponent greater than or equal to the exponents with which they appear in the factorizations of x and y. To obtain the *least* common multiple, we take the exponents to only the highest degree that they appear in those factorizations.
A convenient way to find the LCM is to create a Venn Diagram of the factorizations of the two numbers, including common factors in the overlap, and then taking the product of the factors that appear to be the LCM. To find the LCM of 1200 and 375 for example, we create a Venn diagram as shown below:



Then, we see that the product

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \cdot 5 = 2^4 \cdot 3 \cdot 5^3 = 6,000$$

is the LCM of 1200 and 375.
The technique of finding the LCM is crucial for working with rational expressions later in the mathematics curriculum.

Number Sense 2.4 Use the inverse relationship between raising to a power and extracting the root of a perfect square integer; for an integer that is not square, determine without a calculator the two integers between which its square root lies and explain why.

Problem: Determine without a calculator which is bigger: $\sqrt{291}$ or 17?

Solution: We list some perfect squares and try to get close to 291:

$$\begin{aligned} 13^2 &= 169 \\ 14^2 &= 196 \\ 15^2 &= 225 \\ 16^2 &= 259 \\ 17^2 &= 289 \\ 18^2 &= 324 \end{aligned}$$

Thus, we have

$$324 > 291 > 289,$$

so that

$$\sqrt{324} > \sqrt{291} > \sqrt{289} \quad \Rightarrow \quad 18 > \sqrt{291} > 17.$$

Therefore $\sqrt{291} > 17$.

Further Explanation: It is important to remember that the symbol "$\sqrt{\ }$" denotes the *nonnegative square root* of a number (also referred to as the *principle square root*). Thus, for example $\sqrt{36} = 6$. On the other hand, when we solve $x^2 = 36$, we get

$$x = \pm\sqrt{36} = \pm 6.$$

Number Sense 2.5 Understand the meaning of the absolute value of a number; interpret the absolute value as the distance of the number from zero on a number line; and determine the absolute value of real numbers.

Problem: Consider two numbers A and B on the number line. Determine which is larger: the distance between A and B or the distance between $|A|$ and $|B|$? Always? Sometimes? Never?

Solution: The distance $d(A, B)$ between A and B on a number line can be found with the absolute value:

$$d(A, B) = |A - B| = |B - A|.$$

Since $A - B = -(B - A)$ the order in which we write A and B doesn't matter.
Examples will help illustrate the general idea. If $A = 7$ and $B = 13$, then $|A - B| = |7 - 13| = |-6| = 6$, while

$$||A| - |B|| = ||7| - |13|| = |7 - 13| = |-6| = 6,$$

so that $|A - B| = ||A| - |B||$ in this case.
However, if $A = 7$ and $B = -13$, then

$$|A - B| = |7 - (-13)| = |7 + 13| = |20| = 20,$$

while

$$||A| - |B|| = ||7| - |-13|| = |7 - 13| = 6.$$

In this case, $|A - B| > ||A| - |B||$.
It appears that sometimes $|A - B|$ is greater than $||A| - |B||$, and sometimes they are equal.

Further Explanation: The absolute value of a real number k represents the distance from 0 to k on a number line. If $k > 0$, say if $k = 5.5$ for example, then $|5.5| = 5.5$, so that $|k| = k$. On the other hand, if $k < 0$, say $k = -10$, then $|-10| = 10$, so that $|k| = -k$ (here the "$-$" can be thought of as "the opposite of k", rather than "negative k.") Also, $|0| = 0$.

This leads to the formula definition of the absolute value,

$$|k| = \begin{cases} k & \text{if } k \geq 0 \\ -k & \text{if } k < 0 \end{cases}$$

Algebra and Functions 1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).

Problem: Gabriel bought a CD player, listed at \$$a$, at a 20% discount; he also had to pay an 8% sales tax. After three months he decided that its sound quality was not good enough for his taste, and he sold it in the secondhand market for \$$b$, which was 65% of what he paid originally. Express b as a function of a.

Solution: Since he bought the CD player at a 20% discount, it was priced at $\$(1 - .20)a = \$.8a$. Adding an 8% sales tax is the same as multiplying the price by $1 + .08 = 1.08$, so that he paid $\$(1.08)(.8a) = \$.864a$. Finally, he sold the player for 65% of what he originally paid for it, which means

$$b = \$(0.65)(.864a) = \$.5616a.$$

Algebra and Functions 1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).
Algebra and Functions 4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Problem: A car goes 45 mph and travels 200 miles. How many hours will it take for the car to reach its destination?

Solution: Distance is the product of rate and time, or in symbols, $d = r \times t$. In this case, we have

$$200\,\text{mi} = (45\,\text{mph})t,$$

so that

$$t = \frac{200\,\text{mi}}{45\,\text{mph}} = \frac{200\,\text{mi}}{45\,\frac{\text{mi}}{\text{hr}}} = \frac{200\,\text{mi} \times\ \text{hr}}{45\,\text{mi}} = \frac{40}{9}\,\text{hr}.$$

Thus, it takes $4\frac{4}{9}$ hours to reach its destination.

Further Explanation: To convert the fractional part to minutes, we have

$$\frac{4}{9}\,\text{hr} = \frac{4}{9}\,\text{hr} \times \frac{60\,\text{min}}{\text{hr}} = \frac{240}{9}\,\text{min} = 26\frac{2}{3}\,\text{min}.$$

Therefore, the trip takes 4 hours and $27\frac{2}{3}$ minutes.
For seconds, notice that $\frac{2}{3}\,\text{min} = 40\,\text{s}$, so the trip takes 4 hours, 26 minutes, and 40 seconds.

Algebra and Functions 1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).
Algebra and Functions 4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Problem: A plane flying at 450 mph leaves San Francisco. One-half hour later a second plane flying at 600 mph leaves, flying in the same direction. How long will it take the second plane to catch the first? How far from San Francisco will this event happen?

Solution: If we let d_1 and d_2 represent the distances the planes travel, and t represent the time in hours the *second* plane has traveled, then we are looking for the amount of time t for which $d_1 = d_2$. Since the first plane leaves a half-hour earlier than the second, it has traveled for $t + 0.5$ hours if the second plane has traveled for t hours. Therefore, for the first plane, we have $d_1 = 450(t + 0.5)$, since distance is the product of speed and time. For the second plane, we have $d_2 = 600t$. Setting these equal will give us the time t the second plane has traveled before they meet:

$$\begin{aligned} d_1 &= d_2 \\ 450(t + 0.5) &= 600t \\ 450t + 225 &= 600t \\ 225 &= 150t \\ t &= 1.5 \end{aligned}$$

Therefore, the second plane has traveled for 1.5 hours before catching up to the first. Since both planes traveled the same distance, we can use either d_1 or d_2 to find the distance at which this occurs. If we use d_1, then we have

$$d_1 = 450\,\text{mph}(1.5 + 0.5\,\text{hr}) = 450\,\text{mph}(2\,\text{hr}) = 900\,\text{mi}.$$

The event occurred 900 miles from San Francisco.

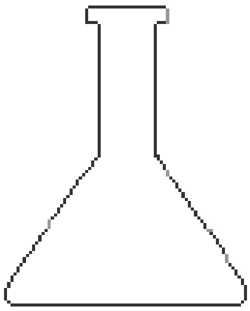
Further Explanation: Of course, the previous problem can be done with the variable t representing the time the first plane left. In that case, the equation to be solved becomes

$$450t = 600(t - 0.5),$$

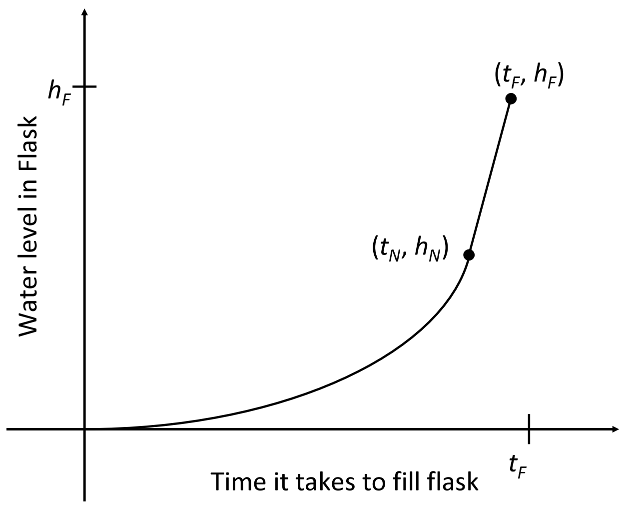
since in that case the second plane has traveled for $t - 0.5$ hours.

Algebra and Functions 1.5 Represent quantitative relationships graphically and interpret the meaning of a specific part of a graph in the situation represented by the graph.

Problem: Water is poured at a constant rate into a flask shaped like the one in the illustration that follows. Draw a graph of the water level in the flask as a function of time.



Solution: The bottom of the flask has much more volume than the the part close to the neck. Consequently, as it is filled with water the water level will rise slowly at first but rise faster and faster until it reaches the neck. Since the neck is a cylinder, the water level in the neck will rise at a constant rate.
This is reflected in the graph below. The water level rises slowly at first, but then faster, until the graph reaches the point (t_N, h_N), where t_N is the time to reaching the neck and h_N the water level at which the neck is reached. After that point, the water level rises at a constant rate, so the graph of the function is linear until reaching the point (t_F, h_F), where t_F is the time to fill the flask, and h_F is the height of the flask.



Algebra and Functions 2.1 Interpret positive whole-number powers as repeated multiplication and negative whole-number powers as repeated division or multiplication by the multiplicative inverse. Simplify and evaluate expressions that include exponents.
Algebra and Functions 2.2 Multiply and divide monomials; extend the process of taking powers and extracting roots to monomials when the latter results in a monomial with an integer exponent.

Problem: Simplify to a monomial, or reduce to a single monomial.

1. $\dfrac{x^5}{x^3}$
2. $\dfrac{x^3}{x^5}$
3. $\dfrac{x^5}{x^5}$
4. $\dfrac{42a^5b^3}{14a^2b^9}$
5. $\dfrac{x^{-8}}{x^{-7}}$
6. $\dfrac{a^7b^{-3}c^9}{a^4b^{-3}c^{10}}$

Solution: We present two solution methods for each.

1. First, $\dfrac{x^5}{x^3} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \dfrac{x}{1} \cdot \dfrac{x}{1} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} = \dfrac{x}{1} \cdot \dfrac{x}{1} \cdot 1 \cdot 1 \cdot 1 = \dfrac{x^2}{1} = x^2.$

   Or, $\dfrac{x^5}{x^3} = x^{5-3} = x^2$, by rules of exponents.

2. First, $\dfrac{x^3}{x^5} = \dfrac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{1}{x} \cdot \dfrac{1}{x} = \cdot 1 \cdot 1 \cdot 1 \dfrac{1}{x} \cdot \dfrac{1}{x} = \dfrac{1}{x^2}.$

   Or, $\dfrac{x^3}{x^5} = x^{3-5} = x^{-2} = \dfrac{1}{x^2}.$

3. First, $\dfrac{x^5}{x^5} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} \cdot \dfrac{x}{x} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1.$

   Or, $\dfrac{x^5}{x^5} = x^{5-5} = x^0 = 1$, by rules of exponents.

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4, First, $\dfrac{42a^5b^3}{14a^2b^9} = \dfrac{2\cdot 3\cdot 7\cdot a\cdot a\cdot a\cdot a\cdot a\cdot b\cdot b\cdot b}{2\cdot 7\cdot a\cdot a\cdot b\cdot b\cdot b\cdot b\cdot b\cdot b\cdot b\cdot b\cdot b} = \dfrac{3a^3}{b^6}$.

Or, $\dfrac{42a^5b^3}{14a^2b^9} = 3a^{5-2}b^{3-9} = 3a^3b^{-6} = \dfrac{3a^3}{b^6}$.

5. First, $\dfrac{x^{-8}}{x^{-7}} = \dfrac{\frac{1}{x^8}}{\frac{1}{x^7}} = \dfrac{1}{x^8} \times \dfrac{\frac{x^7}{1}}{\frac{x^7}{1}} = \dfrac{\frac{x^7}{x^8}}{\frac{x^7}{x^7}} = \dfrac{x^7}{x^8} = \dfrac{1}{x}$.

Or, $\dfrac{x^{-8}}{x^{-7}} = x^{-8-(-7)} = x^{-1} = \dfrac{1}{x}$.

6. First, $\dfrac{a^7b^{-3}c^9}{a^4b^{-3}c^{10}} = \dfrac{a^7\cdot\frac{1}{b^3}\cdot c^9}{a^4\cdot\frac{1}{b^3}\cdot c^{10}} = \dfrac{a^7b^3c^9}{a^4b^3c^{10}} = \dfrac{a^3}{c}$.

Or, $\dfrac{a^7b^{-3}c^9}{a^4b^{-3}c^{10}} = a^{7-4}b^{-3-(-3)}c^{9-10} = a^3b^0c^{-1} = \dfrac{a^3}{c}$.

Further Explanation: Each of the solutions above serves to illustrate the origin of the common rules of exponents that all students should be familiar with, for example,

$$x^m \cdot x^n = x^{m+n} \quad \text{and} \quad \frac{x^m}{x^n} = x^{m-n}.$$

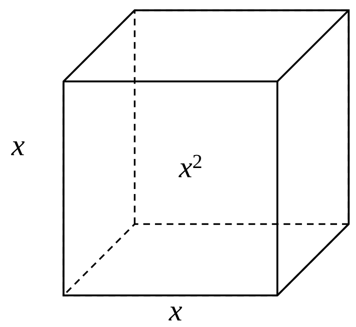
Also, students can reason that $(x^n)^m = x^{nm}$. Moreover, for $x = 0$, we have $x^0 = 1$. Finally, consistent with the rules given above, we take

$$x^{-n} = \frac{1}{x^n}.$$

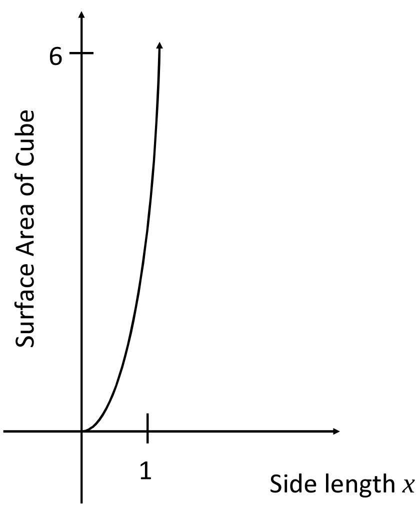
Algebra and Functions 3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.
Algebra and Functions 3.2 Plot the values from the volumes of three-dimensional shapes for various values of the edge lengths (e.g., cubes with varying edge lengths or a triangle prism with a fixed height and an equilateral triangle base of varying lengths).

Problem: Write the equation of the surface area of a cube of side length x. Graph the surface area as a function of x.

Solution: Let y denote the surface area of a cube of side length x. The area of one face of the cube is then x^2. Since a cube has 6 faces, the equation we seek is $y = 6x^2$.



The graph of this equation is a parabola with domain $x > 0$ that passes through the point $(1, 6)$.



Algebra and Functions 3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.

Problem: The amount of paint needed to paint over a surface is directly proportional to the area of the surface. If 2 quarts of paint are needed to paint a square with a side of 3 ft., how many quarts must be purchased to paint a square whose side is 4 ft. 6 in. long?

Solution: Let P represent the number of quarts of paint needed to paint a surface of area A. Then the problem states that $P = kA$, where k is a constant. We can find this constant since it takes 2 quarts of paint to paint a square of side length 3 feet, or of area $A = 9\,\text{ft}^2$. So,

$$\begin{aligned} P &= kA \\ 2 &= k(9) \\ k &= \frac{2}{9} \end{aligned}$$

Therefore, the relationship between the amount of paint and the area can be expressed as $P = \frac{2}{9}A$.
A square with a side of length 4 feet 6 inches, or 4.5 feet, has an area of $4.5^2\,\text{ft}^2 = 20.25\,\text{ft}^2$. To find the amount of paint, we substitute, obtaining

$$\begin{aligned} P &= \frac{2}{9}(4.5 \times 4.5) \\ &= \frac{2}{9}\left(\frac{9}{2} \times \frac{9}{2}\right) \\ &= \frac{9}{2} = 4.5 \end{aligned}$$

Therefore, it requires 4.5 quarts of paint to paint a square with a side of 4 feet 6 inches.

Algebra and Functions 3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the quantities.

Problem: What is the slope of the straight line which is the graph of the function expressing the length of a semicircle as a function of the radius?

Solution: The circumference of a circle is given by the formula

$$C = 2\pi r.$$

A semicircle is half a circle. Therefore, the length S of a semicircle of radius r is half the circumference of the circle of radius r, or $S = \pi r$. The slope of this linear equation is the coefficient of r, or π.

Algebra and Functions 4.1 Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.

Problem: Becky and her sister have some money. The ratio of their money is 3:1. When Becky gives \$5 to her sister, their ratio will be 2:1. How much money does Becky have? (World Math Challenge 1995)

Solution: Let B be the amount of money Becky has, and S the amount of money her sister has. Since initially the ratio of Becky's money to her sister's is 3:1, we have

$$B = 3S$$

After she gives her sister \$5, Becky has $B - 5$ dollars, her sister has $S + 5$ dollars, and since the ratio is now 2:1, we have

$$B - 5 = 2(S + 5)$$

Substituting $3S$ for B in the second equation, we have

$$\begin{aligned} 3S - 5 &= 2(S + 5) \\ 3S - 5 &= 2S + 10 \\ S - 5 &= 10 \end{aligned}$$

or, $S = 15$. Since $B = 3S$ initially, Becky has $B = 3(15) = 45$ dollars.

Algebra and Functions 4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Problem: Three people set out on a car race to see who would be the first to get to town T and back. Anne maintained a steady speed of 80 mph throughout the race. Lee averaged 90 mph on the way out, but he could manage only an average of 70 mph on the way back. Javier started slowly and averaged 65 mph during the first third of the race, but he increased his speed to 85 mph in the second third and finished with a blazing 100 mph in the last third. Who won? (*Note*: This is a difficult problem that would be particularly good for advanced students.)

Solution: Let D represent the distance to town T and back in miles. Let t_A, t_L, and t_J represent the three driver's times (Anne, Lee, and Javier, respectively). Then for Anne's time, we have

$$D = 80t_A \quad \Rightarrow \quad t_A = \frac{D}{80} = 0.0125D.$$

For Lee's time, we use the fact that he traveled $D/2$ miles at 90 mph and $D/2$ miles at 70mph. This means the first half of the trip took him t_{L_1} hours and we have

$$\frac{D}{2} = 90t_{L_1} \quad \Rightarrow \quad t_{L_1} = \frac{D/2}{90} = \frac{D}{180}.$$

Similarly, for the second leg of his trip, we have

$$\frac{D}{2} = 70t_{L_2} \quad \Rightarrow \quad t_{L_2} = \frac{D/2}{70} = \frac{D}{140}.$$

Therefore, Lee's total time was

$$t_L = t_{L_1} + t_{L_2} = \frac{D}{180} + \frac{D}{140} \approx 0.0127D.$$

Finally, similar reasoning shows that Javier's time was

$$t_J = \frac{D/3}{65} + \frac{D/3}{85} + \frac{D/3}{100} = \frac{D}{195} + \frac{D}{255} + \frac{D}{300} \approx 0.0123D.$$

Since for any positive D,

$$0.0123D < 0.0125D < 0.0127D,$$

we have

$$t_J < t_A < t_L,$$

so that Javier won the race.

Measurement and Geometry 1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

Problem: A bucket is put under two faucets. If one faucet is turned on alone, the bucket will be filled in 3 minutes; if the other is turned on, the bucket will be filled in 2 minutes. If both are turned on, how many seconds will it take to fill the bucket?

Solution: If each faucet runs for t minutes, then one faucet fills a fraction of $t/3$ of a bucket in that time, while the other fills a fraction of $t/2$ of a bucket in that time. Working together, they fill a fraction of

$$\frac{t}{3} + \frac{t}{2}$$

of a bucket in t minutes. If we are looking for the time it takes to fill one (whole) bucket, then we can set this expression equal to 1. We have

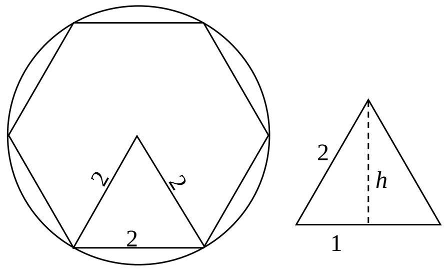
$$\begin{aligned} \frac{t}{3} + \frac{t}{2} &= 1 \\ \frac{5}{6}t &= 1 \\ t &= 6/5, \end{aligned}$$

so that $t = 6/5\,\text{min} = 1.2\,\text{min}$. In other words, it takes 72 seconds to fill the bucket.

Measurement and Geometry 2.1 Use formulas routinely for finding the perimeter and area of basic two-dimensional figures and the surface area and volume of basic three-dimensional figures, including rectangles, parallelograms, trapezoids, squares, triangles, circles, prisms, and cylinders.

Problem: Compute the area and perimeter of a regular hexagon inscribed in a circle of radius 2.

Solution: A picture is shown below.



The measure of a central angle of the hexagon is $60°$. The triangle shown is at least isosceles; therefore the base angles are congruent. In this case, they both measure $60°$, so the triangle is actually equilateral (since equiangular). Thus, the measure of one side of the regular hexagon is 2, so that the perimeter is $6 \cdot 2 = 12$ units.
To find the area, we consider the equilateral triangle and its height h indicated in the diagram. By the Pythagorean theorem, we have

$$h^2 + (1)^2 = 2^2 \quad \Rightarrow \quad h^2 = 3,$$

so that $h = \sqrt{3}$ units. Therefore, the area of the equilateral triangle is

$$A = \frac{1}{2} b \cdot h = \frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3} \text{ units}^2,$$

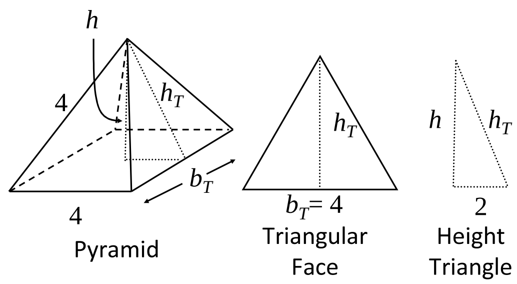
where here we used $b = 2$ units. Since there are six congruent triangles that make up the hexagon, we can find the area of the hexagon:

$$A_{\text{hexagon}} = 6 \cdot \sqrt{3} = 6\sqrt{3} \text{ units}^2.$$

Measurement and Geometry 2.1 Use formulas routinely for finding the perimeter and area of basic two-dimensional figures and the surface area and volume of basic three-dimensional figures, including rectangles, parallelograms, trapezoids, squares, triangles, circles, prisms, and cylinders.

Problem: Compute the volume and surface area of a square-based pyramid whose lateral faces are equilateral triangles with each side equal to 4.

Solution: Shown below is a diagram.



To find the surface area, consider a lateral triangular face of the pyramid. It has base $b_T = 4$ and hypotenuse 4. As in the previous solution, we can find the height of this triangle, h_T:

$$h_T^2 + 2^2 = 4^2 \quad \Rightarrow \quad h_T^2 = 16 - 4 = 12,$$

so that $h_T = \sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$ units. This allows us to compute the area of one triangular face as

$$A_T = \frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3} \text{ units}^2.$$

The base of the pyramid is a square, so it has area $B = 16$ units^2. Since there are 4 triangular faces, we find that the surface area is

$$S.A. = B + 4A_T = 16 + 16\sqrt{3} = 16(1 + \sqrt{3}) \text{ units}^2.$$

The volume of the prism is given by $V = \frac{1}{3}Bh$ where h is the height of the prism. To find the height h, consider the right triangle with base one-half the length of a side, height equal to h, and hypotenuse h_T, as indicated in the picture. We find h:

$$h^2 + 2^2 = (2\sqrt{3})^2 \quad \Rightarrow \quad h^2 = 8,$$

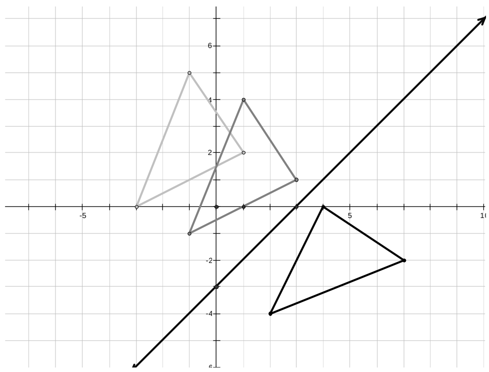
so that $h = 2\sqrt{2}$ units. The volume is therefore

$$V = \frac{1}{3}(16)(2\sqrt{2}) = \frac{32}{3}\sqrt{2} \text{ units}^3.$$

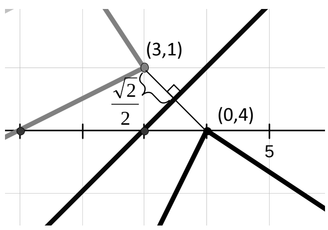
Measurement and Geometry 3.2 Understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, and determine their image under translations and reflections.

Problem: Determine the vertices of a triangle, whose vertices were originally at $(1,2)$, $(-3,0)$, and $(-1,5)$, after it is translated 2 units to the right and 1 unit down and then reflected across the graph of $y = x - 3$.

Solution: In the diagram below, the starting triangle is light grey, the intermediate triangle after translating is dark grey, and the final triangle after reflecting is black.



The translation described has the effect of increasing the x coordinate by 2 units, and decreasing the y unit by 1 unit. Equivalently, $x \mapsto x + 2$ and $y \mapsto y - 1$. Therefore, the intermediate triangle has vertices $(3,1)$, $(-1,-1)$ and $(1,4)$.
The reflection about the line $y = x - 3$ can be described as moving each vertex to a point on the other side of the line but at the same distance from the line. For instance, the point $(3,1)$ is at a distance of $\sqrt{2}/2$ units from the line $y = x - 3$, so we find the point directly across the line at the same distance. This point is $(0,4)$. We find it by moving vertically down to the line and then horizontally to the right to the point $(0,4)$.



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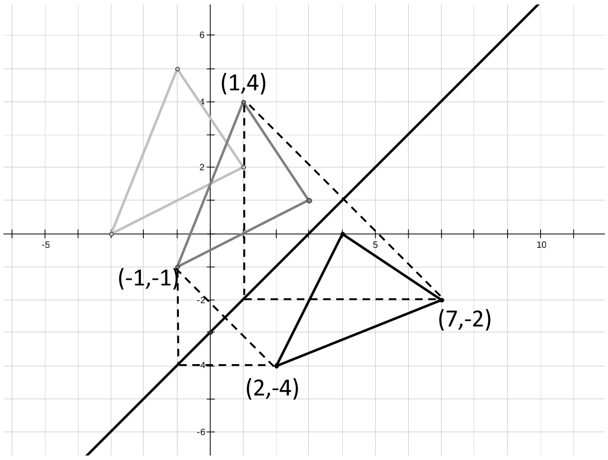
We apply the same reasoning to find the other points of the final triangle.



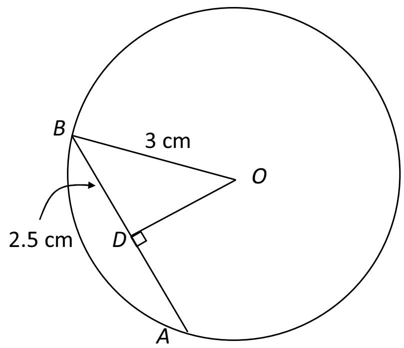
Thus, the final transformation is given by

$$(1,2) \mapsto (3,1) \mapsto (4,0)$$
$$(-3,0) \mapsto (-1,-1) \mapsto (2,-4)$$
$$(-1,5) \mapsto (1,4) \mapsto (7,-2)$$

Measurement and Geometry 3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

Problem: What is the distance from the center of a circle of radius 3 to a chord of length 5 cm?

Solution: Consider the diagram below:



The distance from the center to the chord is the length of segment OD. Since OD is perpendicular to AB, it bisects AB. Therefore, we can apply the Pythagorean theorem to triangle $\triangle DBO$ as follows:

$$(\overline{OD})^2 + (2.5)^2 = 3^2 \quad \Rightarrow \quad (\overline{OD})^2 = 9 - 6.25 = 2.75$$

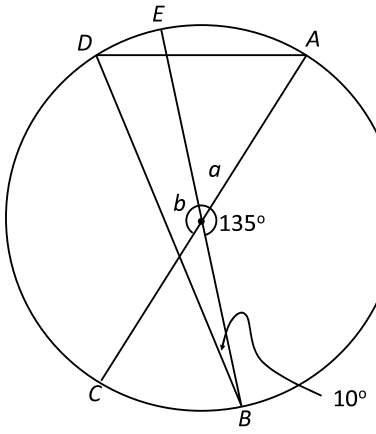
Therefore,

$$\overline{OD} = \sqrt{2.75} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}.$$

Thus, $\overline{OD} \approx 1.66\,\text{cm}$.

Measurement and Geometry 3.1 Identify and construct basic elements of geometric figures (e.g., altitudes, midpoints, diagonals, angle bisectors, and perpendicular bisectors; central angles, radii, diameters, and chords of circles) by using a compass and straightedge.

Problem: Find the missing angles or arcs. ($m\angle B = 10°$.)



Solution: There are several missing pieces of information in the problem:

(1) Minor $\widehat{\text{AB}}$: The measure of minor arc $\widehat{\text{AB}}$ is the same as the measure of the interior angle it subtends. Thus, $m\widehat{\text{AB}} = 135°$.

(2) Angle a: Since $\angle a$ makes a straight line with the angle measuring 135°, we have $m\angle a = 180° - 135° = 45°$.

(3) Major $\widehat{\text{BA}}$: The measure of major arc $\widehat{\text{BA}}$ is the difference of 360° and the measure of the minor arc $\widehat{\text{AB}}$. Thus, $m\widehat{\text{BA}} = 360° - 135° = 225°$.

(4) Angle b: Since $\angle b$ is the vertical angle of the angle measuring 135°, we have $m\angle b = 135°$.

(5) $\widehat{\text{EC}}$: Since $\angle b$ is the central angle which arc $\widehat{\text{EC}}$ subtends, $m\widehat{\text{EC}} = 135°$.

(6) $\widehat{\text{CB}}$: The central angle for arc $\widehat{\text{CB}}$ is a vertical angle to angle $\angle a$, hence congruent to $\angle a$, hence has measure $m\angle a = 45°$. This means $m\widehat{\text{CB}} = 45°$.

(7) $\widehat{\text{ED}}$: The measure of an inscribed angle (an angle inside the circle with vertex on the circle) measures one-half the degree measure of the arc it subtends. Since $\widehat{\text{ED}}$ subtends $\angle B$, and $m\angle B = 10°$, we have $m\widehat{\text{ED}} = 20°$.

(8) $\widehat{\text{DC}}$: Finally, we have $m\widehat{\text{EC}} = 125°$, $m\widehat{\text{ED}} = 20°$, and since $\widehat{\text{DC}} = \widehat{\text{EC}} - \widehat{\text{ED}}$, we have

$$m\widehat{\text{DC}} = m\widehat{\text{EC}} - m\widehat{\text{ED}} = 135° - 20° = 115°.$$